Approximation properties of modified Szász-Mirakyan operators in polynomial weighted space

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Abstract

We introduce certain modified Szász-Mirakyan operators in polynomial weighted spaces of functions of one variable. We studied approximation properties of these operators.

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1. Introduction

Let $J_n^{[\beta]}$ be the Jain operators

$$J_n^{[\beta]}(f,x) = \sum_{k=0}^{\infty} \omega_\beta(k,nx) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \in \mathbf{R}_0 := [0,\infty)$, $n \in \mathbf{N}$, $0 \le \beta < 1$ and

$$\omega_\beta(k,\alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)} \text{ for } \alpha \in \mathbf{R}_0, k \in \mathbf{N}_0 = N \cup \{0\}. \tag{1.2}$$

Approximation properties of $J_n^{[\beta]}$ were examined by Jain [1] for $f \in C(\mathbf{R}_0)$. In the particular case $\beta = 0$, $J_n^{[\beta]}$ turn out to well known the Szász-Mirakyan operators [2]. Kantorovich type extension of the operators (1.1) was discussed in [3]. Various other generalization and its approximation properties of similar type of operators are studied in [4–13]. In this paper, we modify operators $J_n^{[\beta]}$ given by (1.1), i.e. we consider operators

$$J_n^{[\beta]}(f;a_n,b_n;x) = \sum_{k=0}^{\infty} \omega_\beta(k,a_n x) f\left(\frac{k}{b_n}\right), x \in \mathbf{R}_0, n \in \mathbf{N} \tag{1.3}$$

for $f \in C([0,\infty))$, where $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are given increasing and unbounded numerical sequence such that $a_n \ge 1$, $b_n \ge 1$ and $\left(\frac{a_n}{b_n}\right)_1^\infty$ is non decreasing and

$$\frac{a_n}{b_n} = 1 + o\left(\frac{1}{b_n}\right). \tag{1.4}$$

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If $a_n = b_n = n$ for all $n \in \mathbf{N}$, then the operators (1.3) reduce to the operators (1.1).

The paper is organized as follows. In our manuscript, we shall study approximation properties of operators (1.3). In section 2, we shall examine moments of the operators $J_n^{[\beta]}(f; a_n, b_n; x)$. We discuss approximation properties of the operators (1.3) in section 3. We mention Kantorovich type extension of the operators $J_n^{[\beta]}(f; a_n, b_n; x)$ for further research.

2. Moments of $J_n^{[\beta]}(f; a_n, b_n; x)$

In order to obtain moments of $J_n^{[\beta]}(f; a_n, b_n; x)$, we need some background results, which are as follows:

Lemma 1 ([1]). *Let $0 < \alpha < \infty$, $0 \leq \beta < 1$ and let the generalized poisson distribution given by (1.2). Then*

$$\sum_{k=0}^{\infty} \omega_\beta(\alpha, k) = 1. \tag{2.1}$$

Lemma 2 ([1]). *Let $0 < \alpha < \infty$, $0 \leq \beta < 1$. Suppose that*

$$S(r, \alpha, \beta) := \sum_{k=0}^{\infty} (\alpha + \beta k)^{k+r-1} \frac{e^{-(\alpha+\beta k)}}{k!}, r = 0, 1, 2, \ldots$$

and

$$\alpha S(0, \alpha, \beta) := 1.$$

Then

$$S(r, \alpha, \beta) = \alpha S(r-1, \alpha, \beta) + \beta S(r, \alpha + \beta, \beta). \tag{2.2}$$

Also,

$$S(r, \alpha, \beta) = \sum_{k=0}^{\infty} \beta^k (\alpha + k\beta) S(r-1, \alpha + k\beta, \beta). \tag{2.3}$$

From (2.2) and (2.3), when $0 \leq \beta < 1$, we get

$$S(1, \alpha, \beta) = \frac{1}{1-\beta}; \tag{2.4}$$

$$S(2, \alpha, \beta) = \frac{\alpha}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3}; \tag{2.5}$$

$$S(3, \alpha, \beta) = \frac{\alpha^2}{(1-\beta)^3} + \frac{\alpha\beta^2}{(1-\beta)^4} + \frac{\beta^3 + 2\beta^4}{(1-\beta)^5}; \tag{2.6}$$

$$S(4, \alpha, \beta) = \frac{\alpha^3}{(1-\beta)^4} + \frac{6\alpha^2\beta^2}{(1-\beta)^5} + \frac{4\alpha\beta^3 + 11\alpha\beta^4}{(1-\beta)^6} + \frac{\beta^4 + 8\beta^5 + 6\beta^6}{(1-\beta)^7}. \tag{2.7}$$

In the following lemma, we have computed moments up to 4^{th} order.

Lemma 3. *Let $0 \leq \beta < 1$, then the following equalities hold:*

1. $J_n^{[\beta]}(1; a_n, b_n; x) = 1$;

2. $J_n^{[\beta]}(t;a_n,b_n;x)=\dfrac{a_n x}{b_n(1-\beta)}$;
3. $J_n^{[\beta]}(t^2;a_n,b_n;x)=\dfrac{x^2a_n^2}{(1-\beta)^2b_n^2}+\dfrac{xa_n}{(1-\beta)^3b_n^2}$;
4. $J_n^{[\beta]}(t^3;a_n,b_n;x)=\dfrac{x^3a_n^3}{(1-\beta)^3b_n^3}+\dfrac{3x^2a_n^2}{(1-\beta)^4b_n^3}+\dfrac{x(1+2\beta)a_n}{(1-\beta)^5b_n^3}$;
5. $J_n^{[\beta]}(t^4;a_n,b_n;x)=\dfrac{x^4a_n^4}{(1-\beta)^4b_n^4}+\dfrac{6x^3a_n^3}{(1-\beta)^5b_n^4}+\dfrac{x^2(7+8\beta)a_n^2}{(1-\beta)^6b_n^4}+\dfrac{x\left(1+8\beta+6\beta^2\right)a_n}{(1-\beta)^7b_n^4}$.

Proof: Using equalities (2.1), (2.4) to (2.7) and by simple commutation, we obtain

$$\begin{aligned}
J_n^{[\beta]}(1;a_n,b_n;x) &= \sum_{k=0}^{\infty}\omega_\beta(k,a_nx)=1;\\
J_n^{[\beta]}(t;a_n,b_n;x) &= \frac{a_nx}{b_n}\sum_{k=0}^{\infty}\frac{1}{k!}(a_nx+k\beta+\beta)^k e^{-(a_nx+k\beta+\beta)}\\
&= \frac{a_nx}{b_n}S(1,a_nx+\beta,\beta)\\
&= \frac{a_nx}{b_n(1-\beta)};\\
J_n^{[\beta]}(t^2;a_n,b_n;x) &= \sum_{k=0}^{\infty}\frac{a_nx}{k!}(a_nx+k\beta)^{k-1}e^{-(a_nx+k\beta)}\frac{k^2}{b_n^2}\\
&= \frac{a_nx}{b_n^2}\left[S(1,a_nx+\beta,\beta)+S(2,a_nx+2\beta,\beta)\right]\\
&= \frac{a_nx}{b_n^2}\left[\frac{1}{1-\beta}+\frac{a_nx+2\beta}{(1-\beta)^2}+\frac{\beta^2}{(1-\beta)^3}\right]\\
&= \frac{x^2a_n^2}{(1-\beta)^2b_n^2}+\frac{xa_n}{(1-\beta)^3b_n^2};\\
J_n^{[\beta]}(t^3;a_n,b_n;x) &= \sum_{k=0}^{\infty}\frac{a_nx}{k!}(a_nx+k\beta)^{k-1}e^{-(a_nx+k\beta)}\frac{k^3}{b_n^3}\\
&= \frac{a_nx}{b_n^3}\left[S(1,a_nx+\beta,\beta)+3S(2,a_nx+2\beta,\beta)+S(3,a_nx+3\beta,\beta)\right]\\
&= \frac{x^3a_n^3}{(1-\beta)^3b_n^3}+\frac{3x^2a_n^2}{(1-\beta)^4b_n^3}+\frac{x(1+2\beta)a_n}{(1-\beta)^5b_n^3};\\
J_n^{[\beta]}(t^4;a_n,b_n;x) &= \sum_{k=0}^{\infty}\frac{a_nx}{k!}(a_nx+k\beta)^{k-1}e^{-(a_nx+k\beta)}\frac{k^4}{b_n^4}\\
&= \frac{a_nx}{b_n^4}\left[S(1,a_nx+\beta,\beta)+7S(2,a_nx+2\beta,\beta)\right.\\
&\quad \left.+6S(3,a_nx+3\beta,\beta)+S(4,a_nx+4\beta,\beta)\right]\\
&= \frac{x^4a_n^4}{(1-\beta)^4b_n^4}+\frac{6x^3a_n^3}{(1-\beta)^5b_n^4}+\frac{x^2(7+8\beta)a_n^2}{(1-\beta)^6b_n^4}+\frac{x\left(1+8\beta+6\beta^2\right)a_n}{(1-\beta)^7b_n^4}.
\end{aligned}$$

Lemma 4. *Let $0\le\beta<1$, then the following equalities hold:*

1. $J_n^{[\beta]}(t-x;a_n,b_n;x)=\left(\dfrac{a_n}{b_n(1-\beta)}-1\right)x$;
2. $J_n^{[\beta]}((t-x)^2;a_n,b_n;x)=x^2\left(\dfrac{a_n}{(1-\beta)b_n}-1\right)^2+\dfrac{xa_n}{(1-\beta)^3b_n^2}$;

3. $J_n^{[\beta]}((t-x)^3;a_n,b_n;x) = x^3\left(\frac{a_n}{(1-\beta)b_n}-1\right)^3 + \frac{3x^2a_n}{b_n^2(1-\beta)^3}\left(\frac{a_n}{(1-\beta)b_n}-1\right) + \frac{xa_n(1+2\beta)}{(1-\beta)^5b_n^3}$;

4. $J_n^{[\beta]}((t-x)^4;a_n,b_n;x) = x^4\left(\frac{a_n}{(1-\beta)b_n}-1\right)^4 + \frac{6a_nx^3}{(1-\beta)^3b_n^2}\left(\frac{a_n}{(1-\beta)b_n}-1\right)^2$
$$+\frac{a_nx^2}{(1-\beta)^5b_n^3}\left(\frac{a_n(7+8\beta)}{(1-\beta)b_n}-4-8\beta\right)+x\left(\frac{a_n(1+8\beta+6\beta^2)}{(1-\beta)^7b_n^4}\right).$$

Proof of the above lemma, follows from the linearity of the operators $J_n^{[\beta]}(f;a_n,b_n;x)$.

By equality (1.4) and $\lim\limits_{n\to\infty}\beta_n = 0$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} b_nJ_n^{[\beta_n]}(t-x;a_n,b_n;x) &= 0;\\
\lim_{n\to\infty} b_nJ_n^{[\beta_n]}((t-x)^2;a_n,b_n;x) &= x;\\
\lim_{n\to\infty} b_nJ_n^{[\beta_n]}((t-x)^3;a_n,b_n;x) &= 0;\\
\lim_{n\to\infty} b_n^2J_n^{[\beta_n]}((t-x)^4;a_n,b_n;x) &= 3x^2,
\end{aligned}$$

for every $x\in\mathbf{R}_0$.

3. Approximation properties

Let $p\in\mathbf{N}_0$,

$$\omega_0(x)=1,\qquad \omega_p(x)=(1+x^p)^{-1}\quad if\quad p\geq 1,$$

for $x\in\mathbf{R}_0$, and B_p be the set of all functions $f:\mathbf{R}_0\to\mathbf{R}$ for which $f\omega_p$ is bounded on $\mathbf{R}_0$ and the norm is given by the following formula:

$$\|f\|_p=\sup_{x\in\mathbf{R}_0}\omega_p(x)\left|f(x)\right|.$$

Moreover, C_p be the set of all $f\in B_p$ for which $f\omega_p$ is a uniformly continuous function on $\mathbf{R}_0$. The spaces B_p and C_p are called polynomial weighted spaces.

Lemma 5. *Let $r\in\mathbf{N}$ be fixed number. Then there exists positive numerical coefficients $\lambda_{r,j,\beta}$, $1\leq j\leq r$, depending only on r and j such that*

$$J_n^{[\beta]}(t^r;a_n,b_n;x)=\frac{1}{b_n^r(1-\beta)^r}\sum_{j=1}^{r}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}(a_nx)^j,$$

for all $x\in\mathbf{R}_0$ and $n\in\mathbf{N}$. Moreover, we have $\lambda_{r,1,\beta}=1=\lambda_{r,r,\beta}$.

The proof follows by a mathematical induction argument.

Lemma 6. *For given $p\in\mathbf{N}_0$ and $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ there exists a positive constant $M_1(b_1,p,\beta)$ such that*

$$\left\|J_n^{[\beta]}\left(\frac{1}{\omega_p(t)};a_n,b_n;\cdot\right)\right\|_p\leq M_1(b_1,p,\beta),\qquad n\in\mathbf{N}. \tag{3.1}$$

Moreover, for every $f \in C_p$, we have

$$\|J_n^{[\beta]}(f;a_n,b_n;\cdot)\|_p \le M_1(b_1,p,\beta)\|f\|_p, \qquad n \in \mathbf{N}. \tag{3.2}$$

The formula (1.2), (1.3) and the inequality (3.2), show that $J_n^{[\beta]}$, $n \in \mathbf{N}$ is a positive linear operator from the space C_p into C_p, $p \in \mathbf{N}_0$.

Proof: If $p = 0$, then $\left\| J_n^{[\beta]}\left(\frac{1}{\omega_0(t)};a_n,b_n;\cdot\right)\right\|_0 = \sup_{x\in\mathbf{R}_0} |J_n^{[\beta]}(1;a_n,b_n;x)| = 1.$
If $p \ge 1$, then by (1.3), (1.4), Lemma 3 and Lemma 5, we get

$$\begin{aligned}
\omega_p(x)J_n^{[\beta]}\left(\frac{1}{\omega_p(t)};a_n,b_n;x\right) &= \omega_p(x)\left\{1 + J_n^{[\beta]}(t^p;a_n,b_n;x)\right\} \\
&= \frac{1}{1+x^p}\left\{1 + \frac{1}{b_n^p(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}(a_nx)^j\right\} \\
&= \frac{1}{1+x^p} + \frac{1}{(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}\frac{1}{b_n^{p-j}}\left(\frac{a_n}{b_n}\right)^j\frac{x^j}{1+x^p} \\
&\le 1 + \frac{1}{(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{r,j,\beta}}{(1-\beta)^{j-1}}\frac{1}{b_1^{p-j}} = M_1(b_1,p,\beta),
\end{aligned}$$

for all $x \in \mathbf{R}_0$ and $n \in \mathbf{N}$. From this, (3.1) follows.

By (1.3) and definition of norm, we have

$$\|J_n^{[\beta]}(f;a_n,b_n;\cdot)\|_p \le \|J_n^{[\beta]}(\frac{1}{\omega_p(t)};a_n,b_n;\cdot)\|_p\|f\|_p,$$

for every $f \in C_p$, $p \in \mathbf{N}$ and $n \in \mathbf{N}$. From (3.1), the inequalities (3.2) is achieved.

Theorem 1. *For every $p \in \mathbf{N}_0$ there exists a positive constant $M_2(b_1,p,\beta)$ such that*

$$\omega_p(x)J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_p(t)};a_n,b_n;x\right) \le M_2(b_1,p,\beta)\left[x^2\left(\frac{a_n}{(1-\beta)b_n}-1\right)^2 + \frac{x}{(1-\beta)^3b_n}\right], \tag{3.3}$$

for all $x \in \mathbf{R}_0$ and $n \in \mathbf{N}$.

Proof: If $p = 0$, then (3.3) follows from values of $J_n^{[\beta]}\left((t-x)^2;a_n,b_n;x\right)$.
Let $J_n^{[\beta]}(f;x) = J_n^{[\beta]}(f;a_n,b_n;x)$. Notice that

$$J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_p(t)};x\right) = J_n^{[\beta]}\left((t-x)^2;x\right) + J_n^{[\beta]}\left(t^p(t-x)^2;x\right). \tag{3.4}$$

For $p = 1$, we get

$$\begin{aligned}
J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_1(t)};x\right) &= J_n^{[\beta]}\left((t-x)^2;x\right) + J_n^{[\beta]}\left(t(t-x)^2;x\right) \\
&= J_n^{[\beta]}\left((t-x)^2;x\right) + J_n^{[\beta]}\left((t-x)^3;x\right) + xJ_n^{[\beta]}\left((t-x)^2;x\right) \\
&= (1+x)J_n^{[\beta]}\left((t-x)^2;x\right) + J_n^{[\beta]}\left((t-x)^3;x\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
(1+x)J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_1(t)};x\right) &= x^2\left(\frac{a_n}{(1-\beta)b_n}-1\right)^2+\frac{xa_n}{(1-\beta)^3b_n^2}+\frac{x^3}{1+x}\left(\frac{a_n}{(1-\beta)b_n}-1\right)^3\\
&\quad+\frac{3x^2a_n}{(1+x)b_n^2(1-\beta)^3}\left(\frac{a_n}{(1-\beta)b_n}-1\right)+\frac{xa_n(1+2\beta)}{(1+x)(1-\beta)^5b_n^3}\\
&\le M_2(b_1,p,\beta)\left[x^2\left(\frac{a_n}{(1-\beta)b_n}-1\right)^2+\frac{x}{(1-\beta)^3b_n}\right].
\end{aligned}
$$

If $p\geq 2$, then by Lemma 5, we get

$$
\begin{aligned}
\omega_p(x)J_n^{[\beta]}\left(t^p(t-x)^2;x\right) &= \omega_p(x)\left\{J_n^{[\beta]}\left(t^{p+2};x\right)-2xJ_n^{[\beta]}\left(t^{p+1};x\right)+x^2J_n^{[\beta]}\left(t^p;x\right)\right\}\\
&= \frac{x}{b_n(1-\beta)}\left\{\frac{1}{b_n^{p+1}(1-\beta)^{p+1}}\sum_{j=1}^{p+1}\frac{\lambda_{p+2,j,\beta}}{(1-\beta)^{j-1}}a_n^j\frac{x^{j-1}}{1+x^p}\right.\\
&\quad-\frac{2}{b_n^p(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{p+1,j,\beta}}{(1-\beta)^{j-1}}a_n^j\frac{x^j}{1+x^p}\\
&\quad\left.+\frac{1}{b_n^{p-1}(1-\beta)^{p-1}}\sum_{j=1}^{p-1}\frac{\lambda_{p,j,\beta}}{(1-\beta)^{j-1}}a_n^j\frac{x^{j+1}}{1+x^p}\right\}+\frac{1}{(1-\beta)^{2p+3}}\left(\frac{a_n}{b_n}\right)^{p+2}\frac{x^{p+2}}{1+x^p}\\
&\quad-\frac{2}{(1-\beta)^{2p+1}}\left(\frac{a_n}{b_n}\right)^{p+1}\frac{x^{p+2}}{1+x^p}+\frac{1}{(1-\beta)^{2p-1}}\left(\frac{a_n}{b_n}\right)^{p}\frac{x^{p+2}}{1+x^p}\\
&= \frac{x}{b_n(1-\beta)}\left\{\frac{1}{b_n^{p+1}(1-\beta)^{p+1}}\sum_{j=1}^{p+1}\frac{\lambda_{p+2,j,\beta}}{(1-\beta)^{j-1}}a_n^j\frac{x^{j-1}}{1+x^p}\right.\\
&\quad-\frac{2}{b_n^p(1-\beta)^p}\sum_{j=1}^{p}\frac{\lambda_{p+1,j,\beta}}{(1-\beta)^{j-1}}a_n^j\frac{x^j}{1+x^p}\\
&\quad\left.+\frac{1}{b_n^{p-1}(1-\beta)^{p-1}}\sum_{j=1}^{p-1}\frac{\lambda_{p,j,\beta}}{(1-\beta)^{j-1}}a_n^j\frac{x^{j+1}}{1+x^p}\right\}\\
&\quad+\frac{x^{p+2}}{1+x^p}\left(\frac{a_n}{b_n}\right)^p\frac{1}{(1-\beta)^{2p-1}}\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2.
\end{aligned}
$$

Since $0\leq\dfrac{a_n}{b_n}\leq 1$ for $n\in\mathbf{N}$, $(1-\beta)^{-1}\leq(1-\beta)^{-3}$, we have

$$
\begin{aligned}
\omega_p(x)J_n^{[\beta]}\left(t^p(t-x)^2;x\right) &\le \frac{x}{b_n(1-\beta)^3}\left\{\sum_{j=1}^{p+1}\frac{\lambda_{p+2,j,\beta}}{b_1^{p-j+1}(1-\beta)^{p+j}}+2\sum_{j=1}^{p}\frac{\lambda_{p+1,j,\beta}}{b_1^{p-j}(1-\beta)^{p+j-1}}\right.\\
&\quad\left.+\sum_{j=1}^{p-1}\frac{\lambda_{p,j,\beta}}{b_1^{p-j-1}(1-\beta)^{p+j-2}}\right\}+\frac{x^2}{(1-\beta)^{2p-1}}\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2.\\
&\le M_2(b_1,p,\beta)\left\{x^2\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2+\frac{x}{b_n(1-\beta)^3}\right\}. \qquad (3.5)
\end{aligned}
$$

for $x\in\mathbf{R}_0$, $n\in\mathbf{N}$. Using (3.5) in (3.4), we obtain (3.3) for $p\geq 2$.

Thus, the proof is completed.

Now, we approximate $J_n^{[\beta]}(f;a_n,b_n;x)$ using the modulus of continuity $\omega_1(f,C_p)$ and the modulus of smoothness $\omega_2(f,C_p)$ of function $f\in C_p$, $p\in\mathbf{N}_0$

$$\omega_1(f,C_p,t):=\sup_{0\le h\le t}\|\triangle_h f(\cdot)\|_p,\qquad \omega_2(f,C_p,t):=\sup_{0\le h\le t}\|\triangle_h^2 f(\cdot)\|_p,$$

for $t\ge 0$, where

$$\triangle_h f(x)=f(x+h)-f(x),\qquad \triangle_h^2 f(x)=f(x)-2f(x+h)+f(x+2h).$$

Let

$$\xi_{n,\beta}(x)=x^2\left(\frac{a_n}{b_n(1-\beta)}-1\right)^2+\frac{x}{b_n(1-\beta)^3},\quad x\in\mathbf{R}_0,x\in\mathbf{N}. \tag{3.6}$$

Theorem 2. *Suppose that $f\in C_p^2$ with a fixed $p\in\mathbf{N}_0$. Then there exists a positive constant $M_3(b_1,p,\beta)$ such that*

$$\omega_p(x)|J_n^{[\beta]}(f;a_n,b_n;x)-f(x)|\le\|f'\|_p\left|\frac{a_n}{b_n(1-\beta)}-1\right|x+\|f''\|_pM_3(b_1,p,\beta)\xi_{n,\beta}(x),$$

for all $x\in\mathbf{R}_0$, $n\in\mathbf{N}$.

Proof: Notice that $J_n^{[\beta]}(0;a_n,b_n;x)=f(0),n\in\mathbf{N}$, which implies (3.7) for $x=0$.
Let $x>0$ and let $J_n^{[\beta]}(f;x)=J_n^{[\beta]}(f;a_n,b_n;x)$. For $f\in C_p^2$ and $t\in\mathbf{R}_0$,

$$f(t)=f(x)+f'(x)(t-x)+\int_x^t(t-u)f''(u)du. \tag{3.7}$$

Applying $J_n^{[\beta]}(f;x)$ on both side, we obtain

$$J_n^{[\beta]}(f(t);x)=f(x)+f'(x)J_n^{[\beta]}((t-x);x)+J_n^{[\beta]}\left(\int_x^t(t-u)f''(u)du;x\right).$$

Notice that

$$\left|\int_x^t(t-u)f''(u)du\right|\le\|f''\|_p\left(\frac{1}{\omega_p(t)}+\frac{1}{\omega_p(x)}\right)(t-x)^2.$$

Now, using above inequality, we have

$$\begin{aligned}\omega_p(x)|J_n^{[\beta]}(f(t);x)-f(x)| &\le \|f'\|_pJ_n^{[\beta]}((t-x);x)\\ &\quad+\|f''\|_p\omega_p(x)J_n^{[\beta]}\left(\left(\frac{1}{\omega_p(t)}+\frac{1}{\omega_p(x)}\right)(t-x)^2;x\right)\\ &\le \|f'\|_pJ_n^{[\beta]}((t-x);x)\\ &\quad+\|f''\|_p\left(\omega_p(x)J_n^{[\beta]}\left(\frac{(t-x)^2}{\omega_p(t)};x\right)+J_n^{[\beta]}\left((t-x)^2;x\right)\right).\end{aligned}$$

Now, using (3.3) and (3.6), we get

$$\omega_p(x)|J_n^{[\beta]}(f(t);x)-f(x)|\quad\le\quad\|f'\|_p\left|\frac{a_n}{b_n(1-\beta)}-1\right|x+\|f''\|_p\xi_{n,\beta}(x)M_3(b_1,n,\beta).$$

Thus, the proof is completed.

Corollary 1. *Let $\rho(x) = (1+x^2)^{-1}$, $x \in \mathbf{R}_0$. Suppose that $f \in C_p^2$ with a fixed $p = 2$. Then there exists a positive constant $M_4(b_1, p, \beta)$ such that*

$$\begin{aligned}\|[J_n^{[\beta]}(f; a_n, b_n; x) - f(x)]\ \rho\|_2 \quad &\leq \quad \left(1 - \frac{a_n}{b_n(1-\beta)}\right) \|f'\|_2 \\ &\quad + M_4(b_1, p, \beta)\|f''\|_2 b_n^{-1}(1-\beta)^{-3}, n \in \mathbf{N}\end{aligned} \tag{3.8}$$

Theorem 3. *Suppose that $f \in C_p$ with a fixed $p \in \mathbf{N}_0$. Then there exists a positive constant $M_5(b_1, p, \beta)$ such that*

$$\begin{aligned}\omega_p|J_n^{[\beta]}(f; a_n, b_n; x) - f(x)| \quad &\leq \quad \left|\frac{a_n}{b_n(1-\beta)} - 1\right| x\,(\xi_{n,\beta}(x))^{-1/2}\,\omega_1\left(f; C_p; \sqrt{\xi_{n,\beta}(x)}\right) \\ &\quad + M_5(b_1, p, \beta)\omega_2\left(f; C_p; \sqrt{\xi_{n,\beta}(x)}\right),\end{aligned}$$

for all $x > 0$ and $n \in \mathbf{N}$, where $\xi_{n,\beta}(\cdot)$ is defined in (3.6). For $x = 0$, it follows that $J_n^{[\beta]}(f; a_n, b_n; 0) = f(0)$.

Proof: We shall apply the Steklov function f_h for $f \in C_p$:

$$f_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} [f(x+s+t) - f(x+2(s+t))]\, dsdt,$$

$x \in \mathbf{R}_0$, $h > 0$, for which we have

$$\begin{aligned}f_h'(x) \quad &= \quad \frac{1}{h^2}\int_0^{h/2} \left[8\triangle_{h/2}f(x+s) - 2\triangle_h f(x+2s)\right] ds, \\ f_h''(x) \quad &= \quad \frac{1}{h^2}\left[8\triangle_{h/2}^2 f(x) - \triangle_h^2 f(x)\right].\end{aligned}$$

Hence, for $h > 0$, we have

$$\|f_h - f\|_p \quad \leq \quad \omega_2(f, C_p; h), \tag{3.9}$$

$$\|f_h'\|_p \quad \leq \quad 5h^{-1}\omega_1(f, C_p; h)\frac{\omega_p(x)}{\omega_p(x+h)}, \tag{3.10}$$

$$\|f_h''\|_p \quad \leq \quad 9h^{-2}\omega_2(f, C_p; h), \tag{3.11}$$

which show that $f_h \in C_p^2$ if $f \in C_p$. By denoting $J_n^{[\beta]}(f; a_n, b_n; x)$ by $J_n^{[\beta]}(f; x)$ we can write

$$\begin{aligned}\omega_p(x)|J_n^{[\beta]}(f; x) - f(x)| \quad &\leq \quad \omega_p(x)\left\{|J_n^{[\beta]}(f - f_h; x)| + |J_n^{[\beta]}(f_h; x) - f_h(x)|\right. \\ &\quad \left. + |f_h(x) - f(x)|\right\} := A_1 + A_2 + A_3,\end{aligned}$$

for $x > 0$, $h > 0$ and $n \in \mathbf{N}$. By (3.2) and (3.9), we have

$$\begin{aligned}A_1 \quad &\leq \quad M_1(b_1, p, \beta)\|f - f_h\|_p \leq M_1(b_1, p, \beta)\omega_2(f, C_p; h), \\ A_3 \quad &\leq \quad \omega_2(f, C_p; h).\end{aligned}$$

Applying Theorem 2, inequalities (3.10) and (3.11), we get

$$\begin{aligned} A_2 &\leq \|f'\|_p \left| \frac{a_n}{b_n(1-\beta)} - 1 \right| x + \|f''\|_p M_3(b_1,p,\beta)\xi_{n,\beta}(x) \\ &\leq \omega_1(f,C_p;h)\frac{\omega_p(x)}{\omega_p(x+h)}\frac{5x}{h}\left| \frac{a_n}{b_n(1-\beta)} - 1 \right| + \frac{9}{h^2}\omega_2(f,C_p;h)M_3(b_1,p,\beta)\xi_{n,\beta}(x) \end{aligned}$$

Combining these and setting $h = \sqrt{\xi_{n,\beta}(x)}$, for fixed $x > 0$ and $n \in \mathbf{N}$, we obtain the desired result.

Theorem 4. *Let $f \in C_p$, $p \in \mathbf{N}_0$, and let $\rho(x) = (1+x^2)^{-1}$ for $x \in \mathbf{R}_0$. Then there exists a positive constant $M_6(b_1,p,\beta)$ such that*

$$\begin{aligned} \left\| \left[J_n^{[\beta]}(f;a_n,b_n;x) - f \right] \rho \right\|_p \leq & \left(1 - \frac{a_n}{b_n(1-\beta)}\right)\sqrt{b_n}\ \ \omega_1\left(f, C_p; 1/\sqrt{b_n(1-\beta)^3}\right) \\ & + M_6(b_1,p,\beta)\omega_2\left(f, C_p; 1/\sqrt{b_n(1-\beta)^3}\right), \quad n \in \mathbf{N}. \end{aligned}$$

From Theorems 3 and 4, we derive the following corollary:

Corollary 2. *Let $f \in C_p$, $p \in \mathbf{N}_0$, $\beta_n \to 0$ as $n \to \infty$. Then for $J_n^{[\beta_n]}$ defined by (1.3), we have*

$$\lim_{n\to\infty} J_n^{[\beta_n]}(f;a_n,b_n;x) = f(x), \qquad x \in \mathbf{R}_0. \tag{3.12}$$

Furthermore, the convergence of (3.12) is uniformly on every interval $[x_1, x_2]$, where $x_2 > x_1 \geq 0$.

Remark 1. *The error of approximation of a function $f \in C_p, p \in \mathbf{N}_0$ by $J_n^{[\beta]}(f;a_n,b_n;.)$ where $a_n = n^r + \frac{1}{n}$ and $b_n = n^r, r > 1$ is smaller than by the operators* (1.1).

4. The Operators $K_n^{[\beta]}(f;a_n,b_n)$

In 1985, Umar and Razi [3] introduced Kantorovich type extension of the operators (1.1). Motivated by these, we introduce the further generalization of the operators (1.3), in the following way:

$$K_n^{[\beta]}(f;a_n,b_n;x) = b_n \sum_{k=0}^{\infty} \omega_\beta(k, a_n x) \int_{\frac{k}{b_n}}^{\frac{k+1}{b_n}} f(t)\,dt, \tag{4.1}$$

where $x \in \mathbf{R}_0 := [0,\infty)$, $n \in \mathbf{N}$, $0 \leq \beta < 1$ and $\omega_\beta(k, a_n x)$ as same in (1.3).
It is obvious that, the operators $K_n^{[\beta]}(f,a_n,b_n)$, $n \in \mathbf{N}$, defined in (4.1), we can consider for function $f \in C_p$, $p \in \mathbf{N}_0$. For these operators and $f \in C_p$, we can prove lemma and theorems similar to the operators $J_n^{[\beta]}(f,a_n,b_n)$. One can study properties of $K_n^{[\beta]}(f,a_n,b_n)$ for functions $f \in L(\mathbf{R}_0)$ as same as discussed in [14].

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